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Understanding and proving the expansion by regions

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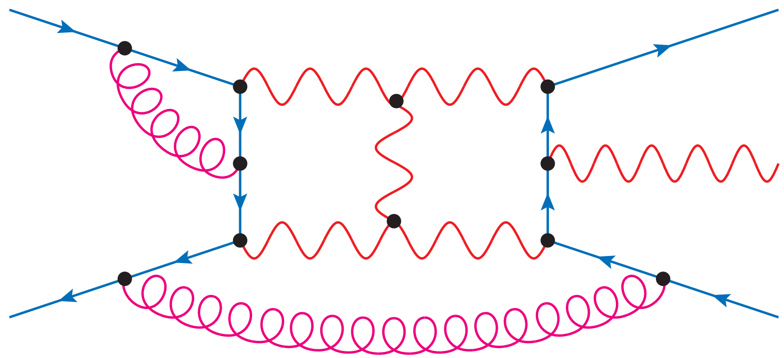
Disclaimer: no practical user's guide to expanding by regions, but demonstration why it works.

I The strategy of regions

Consider complicated (multi-)loop integral:



$$F = \int d^d k_1 \int d^d k_2 \cdots \frac{1}{(k_1+p_1)^2 - m_1^2} \times$$
$$\times \frac{1}{(k_1+k_2+p_2)^2 - m_2^2} \cdots$$

- complicated function of internal masses m_i and kinematical parameters p_i^2, $p_i \cdot p_j$
- exact evaluation often hard or impossible

Exploit parameter hierarchies, e.g. large energies $Q \gg$ small masses m:

$\hookrightarrow$ expand integral in small ratios $\frac{m^2}{Q^2}$

$\hookrightarrow$ simplification if expansion of integrand before integration

But:

⋆ loop-momentum components k_i^μ can take any values (large, small, mixed, ...)

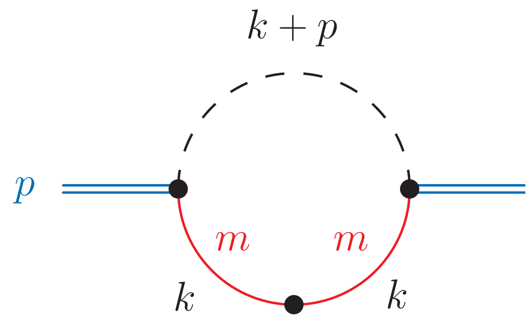
⋆ naive expansions of integrand may generate new singularities

$\hookrightarrow$ Need sophisticated methods of asymptotic expansions.

Simple example: large-momentum expansion

$$F = \int \frac{Dk}{(k+p)^2\,(k^2-m^2)^2}$$

$$\left[\int Dk \equiv \mu^{2\epsilon} e^{\epsilon\gamma_E} \int \frac{d^d k}{i\pi^{d/2}} \qquad d = 4-2\epsilon \right]$$



Large momentum $|p^2| \gg m^2$ $\rightsquigarrow$ expand in $\frac{m^2}{p^2}$.

Integral is UV- and IR-finite, the exact result is known: $[p^2 \to p^2 + i0]$

$$F = \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) + \ln\left(1-\frac{m^2}{p^2}\right)\right] + \mathcal{O}(\epsilon) \xrightarrow[\text{expand}]{} \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{m^2}{p^2}\right)^n\right] + \mathcal{O}(\epsilon)$$

Assume we could not calculate this integral exactly . . .

Expansion by regions

Beneke, Smirnov, Nucl. Phys. B 522, 321 (1998)
Smirnov, Rakhmetov, Theor. Math. Phys. 120, 870 (1999)
Smirnov, Phys. Lett. B 465, 226 (1999)

$\hookrightarrow$ 2 relevant regions:

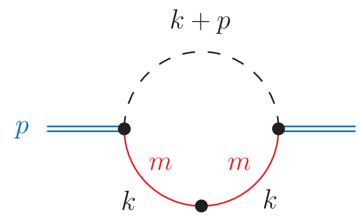
- **hard** (h): $k \sim p \Rightarrow \sum_i T_i^{(h)} \frac{1}{(k^2-m^2)^2} = \sum_{i=0}^{\infty}(1+i)\frac{(m^2)^i}{(k^2)^{2+i}}$
- **soft** (s): $k \sim m \Rightarrow \sum_j T_j^{(s)} \frac{1}{(k+p)^2} = \sum_{j_1,j_2=0}^{\infty} \frac{(j_1+j_2)!}{j_1!\,j_2!}\,\frac{(-2k\cdot p)^{j_1}\,(-k^2)^{j_2}}{(p^2)^{1+j_1+j_2}}$

$\Rightarrow$ Integrate each expanded term over the **whole integration domain.**

$\Rightarrow$ Set scaleless integrals to zero ($\rightsquigarrow$ dimensional regularization).

Large-momentum expansion (2)

$$F = \int \frac{Dk}{(k+p)^2\,(k^2-m^2)^2}$$



Leading-order contributions:

- **hard:** $F_0^{(h)} = \int \frac{Dk}{(k+p)^2\,(k^2)^2} = \frac{1}{p^2}\left(-\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)\left(\frac{\mu^2}{-p^2}\right)^{\epsilon} \rightsquigarrow$ IR-singular!
- **soft:** $F_0^{(s)} = \int \frac{Dk}{p^2\,(k^2-m^2)^2} = \frac{1}{p^2}\left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)\left(\frac{\mu^2}{m^2}\right)^{\epsilon} \rightsquigarrow$ UV-singular!

↪ Contributions are manifestly homogeneous in the expansion parameter $\frac{m^2}{p^2}$.

↪ Singularities are cancelled in the sum of all contributions, exact result approximated:

$$F_0 = F_0^{(h)} + F_0^{(s)} = \frac{1}{p^2}\ln\left(\frac{-p^2}{m^2}\right) + \mathcal{O}(\epsilon) = F + \mathcal{O}\left(\frac{m^2}{(p^2)^2}\right) \quad ✓$$

[The full result F is exactly reproduced when summing the expansion to all orders.]

Questions: Why does this expansion by regions work?

Expansion by regions successfully applied to many complicated (multi-)loop integrals. But:

- Didn't we double-count every $k \in \mathbb{R}^d$ when replacing $\int Dk \to \sum_i \int Dk\, T_i^{(h)} + \sum_j \int Dk\, T_j^{(s)}$?
- What ensures the cancellation of singularities? (IR ↔ UV!)
- Is the chosen set of regions complete?

II Why does it work?

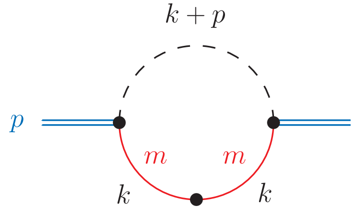
Idea based on a 1-dimensional example from M. Beneke in
Smirnov, *Applied Asymptotic Expansions In Momenta And Masses*

The **expansions** above **converge absolutely** within **domains** D_h, D_s:

(h): $\displaystyle \frac{1}{(k^2-m^2)^2} = \sum_i T_i^{(h)} \frac{1}{(k^2-m^2)^2}$ within $D_h = \left\{ k \in \mathbb{R}^d \middle| |k^2| \geq \Lambda^2 \right\}$,

(s): $\displaystyle \frac{1}{(k+p)^2} = \sum_j T_j^{(s)} \frac{1}{(k+p)^2}$ within $D_s = \left\{ k \in \mathbb{R}^d \middle| |k^2| < \Lambda^2 \right\}$,

with $m^2 \ll \Lambda^2 \ll |p^2| \rightsquigarrow D_h \cup D_s = \mathbb{R}^d,\ D_h \cap D_s = \emptyset.$



The expansions **commute** with integrals restricted to the corresponding domains:

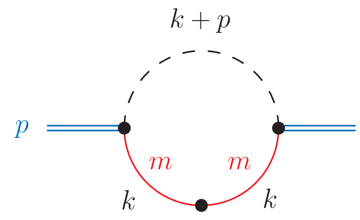
$$F = \int\limits_{k\in\mathbb{R}^d} Dk \underbrace{\frac{1}{(k+p)^2\,(k^2-m^2)^2}}_{I} = \sum_i \int\limits_{k\in D_h} Dk\, T_i^{(h)} I + \sum_j \int\limits_{k\in D_s} Dk\, T_j^{(s)} I$$

$$= \sum_i \Bigg(\int\limits_{k\in\mathbb{R}^d} Dk\, T_i^{(h)} I - \sum_j \int\limits_{k\in D_s} Dk\, T_j^{(s)} T_i^{(h)} I \Bigg) + \sum_j \Bigg(\int\limits_{k\in\mathbb{R}^d} Dk\, T_j^{(s)} I - \sum_i \int\limits_{k\in D_h} Dk\, T_i^{(h)} T_j^{(s)} I \Bigg)$$

$$= \underbrace{\sum_i \int Dk\, T_i^{(h)} I}_{F^{(h)}} + \underbrace{\sum_j \int Dk\, T_j^{(s)} I}_{F^{(s)}} - \underbrace{\sum_{i,j} \int Dk\, T_{i,j}^{(h,s)} I}_{F^{(h,s)}}$$

$$\left[T_i^{(h)} T_j^{(s)} I = T_j^{(s)} T_i^{(h)} I \equiv T_{i,j}^{(h,s)} I \right]$$

Identity: $F = \underbrace{\sum_i \int Dk\, T_i^{(h)} I}_{F^{(h)}} + \underbrace{\sum_j \int Dk\, T_j^{(s)} I}_{F^{(s)}} - \underbrace{\sum_{i,j} \int Dk\, T_{i,j}^{(h,s)} I}_{F^{(h,s)}}$



$\hookrightarrow$ all integrals over $k \in \mathbb{R}^d$ as when expanding by regions $\Rightarrow$ boundary Λ irrelevant.

Additional overlap contribution $F^{(h,s)}$?

$$F^{(h,s)} = \sum_{i=0}^{\infty} (1+i) \sum_{j_1,j_2=0}^{\infty} (-1)^{j_2} \frac{(j_1+j_2)!}{j_1!\, j_2!} \frac{(m^2)^i}{(p^2)^{1+j_1+j_2}} \int Dk \frac{(-2k\cdot p)^{j_1}}{(k^2)^{2+i-j_2}} = 0 \quad \text{scaleless!}$$

$\hookrightarrow$ $\boxed{F = F^{(h)} + F^{(s)}}$ as before, but now obtained without evaluating F, $F^{(h)}$, $F^{(s)}$!

[Actually $\int \frac{Dk}{(k^2)^2} = \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}}$ cancels corresponding singularities in $F^{(h)}$ and $F^{(s)}$.]

Example with 3 regions: threshold expansion for heavy-particle pair production

Here: 3 regions hard (h), soft (s) and potential (p). A similar proof shows:

$$F = F^{(h)} + \underbrace{F^{(s)}}_{=0} + F^{(p)} - \Big(\underbrace{F^{(h,s)}}_{=0} + \underbrace{F^{(h,p)}}_{=0} + \underbrace{F^{(s,p)}}_{=0}\Big) + \underbrace{F^{(h,s,p)}}_{=0 \text{ (scaleless)}}$$

III The general formalism

Consider (multiple) integral $F = \int Dk\, I$ over domain D (e.g. $D = \mathbb{R}^d$) and a set of N regions $R = \{x_1, \ldots, x_N\}$ with, for each region $x \in R$, an expansion $T^{(x)} = \sum_j T_j^{(x)}$ which converges absolutely in the domain $D_x \subset D$.

Conditions:
- $\bigcup_{x \in R} D_x = D, \quad D_x \cap D_{x'} = \emptyset \, \forall x \neq x'$
- expansions commute: $T^{(x)} T^{(x')} I = T^{(x')} T^{(x)} I \equiv T^{(x,x')} I$
- regularization for singularities, e.g. dimensional (+ analytic) reg.

$\hookrightarrow$ Then this **identity** holds (proven iteratively): $\left[F^{(x,\ldots)} \equiv \sum_{j,\ldots} \int Dk\, T_{j,\ldots}^{(x,\ldots)} I\right]$

$$F = \sum_{x'_1 \in R} F^{(x'_1)} - \sum_{\{x'_1, x'_2\} \subset R} F^{(x'_1, x'_2)} + \ldots - (-1)^n \sum_{\{x'_1, \ldots, x'_n\} \subset R} F^{(x'_1, \ldots, x'_n)} + \ldots - (-1)^N F^{(x_1, \ldots, x_N)}$$

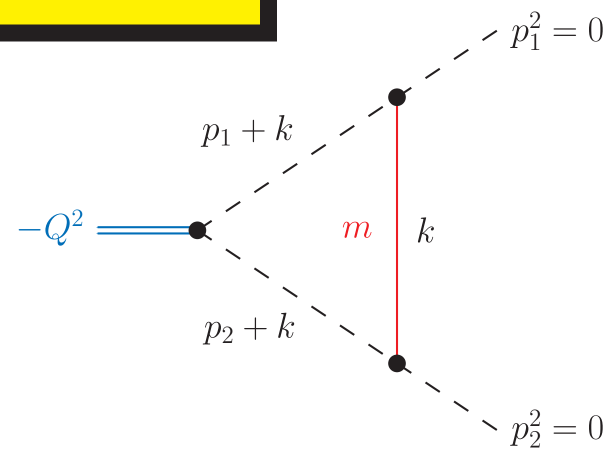
$\hookrightarrow$ exact when expansions are summed to all orders ✓

Usually regions & regularization chosen such that $F^{(x'_1, \ldots, x'_n)} = 0$ (scaleless) for $n \geq 2$.
[$\Longleftrightarrow$ Each $F^{(x)}$ is homogeneous function of expansion parameter and $T^{(x)} I \neq T^{(x')} I \, \forall x \neq x'$.]

If $F^{(x'_1, x'_2, \ldots)} \neq 0 \rightsquigarrow$ relevant overlap contributions ($\rightarrow$ "zero-bin subtractions"), they appear e.g. when avoiding analytic regularization in SCET:

Chiu, Fuhrer, Hoang, Kelley, Manohar, PRD 79, 053007 (2009): "Δ-regulator";
Bauer, Lange, Ovanesyan, arXiv:1010.1072 [hep-ph]; ...

IV Complications & generalization



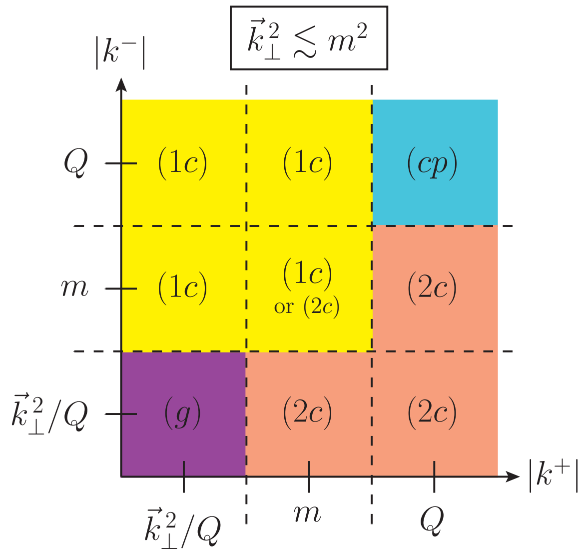
Sudakov form factor: non-commuting expansions

[light-cone coordinates $2p_{1,2} \cdot k = Qk^{\pm}$, $p_{1,2} \cdot k_\perp = 0$]

Sudakov limit: $-(p_1 - p_2)^2 = \boxed{Q^2 \gg m^2}$

Regions & domains:

- **hard** (h): $k^+, k^-, |\vec{k}_\perp| \sim Q \Rightarrow D_h = \left\{ k \in \mathbb{R}^d \middle| \vec{k}_\perp^2 \gg m^2 \right\}$
- **1-collinear** $(1c)$: $k^+ \sim \frac{m^2}{Q}$, $k^- \sim Q$, $|\vec{k}_\perp| \sim m$
- **2-collinear** $(2c)$: $k^+ \sim Q$, $k^- \sim \frac{m^2}{Q}$, $|\vec{k}_\perp| \sim m$
- **Glauber** (g): $k^+, k^- \sim \frac{m^2}{Q}$, $|\vec{k}_\perp| \sim m$ ($\rightsquigarrow$ scaleless)
- **collinear plane** (cp): $k^+, k^- \sim Q$, $|\vec{k}_\perp| \sim m$ ($\rightsquigarrow$ scaleless)



Most expansions commute, but $T^{(g)} T^{(cp)} I \neq T^{(cp)} T^{(g)} I$!

$\hookrightarrow$ Obtain only combinations of regions without both (g) and (cp)

+ extra terms which cancel at the integrand level:

$$F = \sum_{x'_1} F^{(x'_1)} - \sum_{\{x'_1, x'_2\} \neq \{g, cp\}} F^{(x'_1, x'_2)} + \sum_{\{x'_1, x'_2, x'_3\} \not\supset \{g, cp\}} F^{(x'_1, x'_2, x'_3)} - \left(F^{(h,1c,2c,g)} + F^{(h,1c,2c,cp)} \right)$$

V Summary

Expansion by regions for general integrals

- conditions for regions (+ corresponding expansions & domains) established.
- identity proven $\leadsto$ relates exact integral to sum of expanded terms.
- This identity includes overlap contributions which are usually scaleless:

$$F = \sum_{x'_1 \in R} F^{(x'_1)} - \sum_{\{x'_1, x'_2\} \subset R} F^{(x'_1, x'_2)} + \ldots - (-1)^n \sum_{\{x'_1, \ldots, x'_n\} \subset R} F^{(x'_1, \ldots, x'_n)} + \ldots - (-1)^N F^{(x_1, \ldots, x_N)}$$

- successful application to 1-loop examples (setup & check of conditions, evaluation of regions to all orders, comparison to exact result)

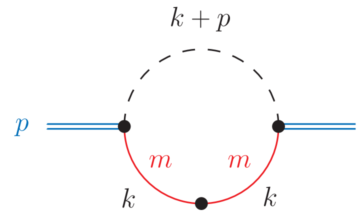
Generalization for non-commuting expansions

- worked out for example of Sudakov form factor
- generalized identity & cancellation of extra terms shown

Extra slides

Large-momentum expansion (3)

$$F = \int \frac{Dk}{(k+p)^2\,(k^2-m^2)^2}$$



Leading-order contributions:

- **hard:** $F_0^{(h)} = \int \frac{Dk}{(k+p)^2\,(k^2)^2} = \frac{1}{p^2}\left(-\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)\left(\frac{\mu^2}{-p^2}\right)^{\epsilon} \rightsquigarrow$ IR-singular!
- **soft:** $F_0^{(s)} = \int \frac{Dk}{p^2\,(k^2-m^2)^2} = \frac{1}{p^2}\left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right)\left(\frac{\mu^2}{m^2}\right)^{\epsilon} \rightsquigarrow$ UV-singular!

↪ Contributions are manifestly homogeneous in the expansion parameter $\frac{m^2}{p^2}$.

↪ Singularities are cancelled in the sum of all contributions, exact result approximated:

$$F_0 = F_0^{(h)} + F_0^{(s)} = \frac{1}{p^2}\ln\left(\frac{-p^2}{m^2}\right) + \mathcal{O}(\epsilon) = F + \mathcal{O}\left(\frac{m^2}{(p^2)^2}\right) \quad \checkmark$$

Expand to all orders in $\frac{m^2}{p^2}$:

$[(\alpha)_n \equiv \Gamma(\alpha+n)/\Gamma(\alpha)]$

$$F^{(h)} = \frac{1}{p^2}\,\frac{e^{\epsilon\gamma_E}\,\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{(-\epsilon)\,\Gamma(1-2\epsilon)}\left(\frac{\mu^2}{-p^2}\right)^{\epsilon}\sum_{i=0}^{\infty}\frac{(2\epsilon)_i}{i!}\left(\frac{m^2}{p^2}\right)^{i} = F_0^{(h)} + \frac{2}{p^2}\ln\left(1-\frac{m^2}{p^2}\right) + \mathcal{O}(\epsilon)$$

$$F^{(s)} = \frac{1}{p^2}\,e^{\epsilon\gamma_E}\,\Gamma(\epsilon)\left(\frac{\mu^2}{m^2}\right)^{\epsilon}\sum_{j=0}^{\infty}\frac{(\epsilon)_j}{(1-\epsilon)_j}\left(\frac{m^2}{p^2}\right)^{j} = F_0^{(s)} - \frac{1}{p^2}\ln\left(1-\frac{m^2}{p^2}\right) + \mathcal{O}(\epsilon)$$

$$\hookrightarrow F = F^{(h)} + F^{(s)} = \frac{1}{p^2}\left[\ln\left(\frac{-p^2}{m^2}\right) + \ln\left(1-\frac{m^2}{p^2}\right)\right] + \mathcal{O}(\epsilon) \quad \checkmark$$

"Real-life" example

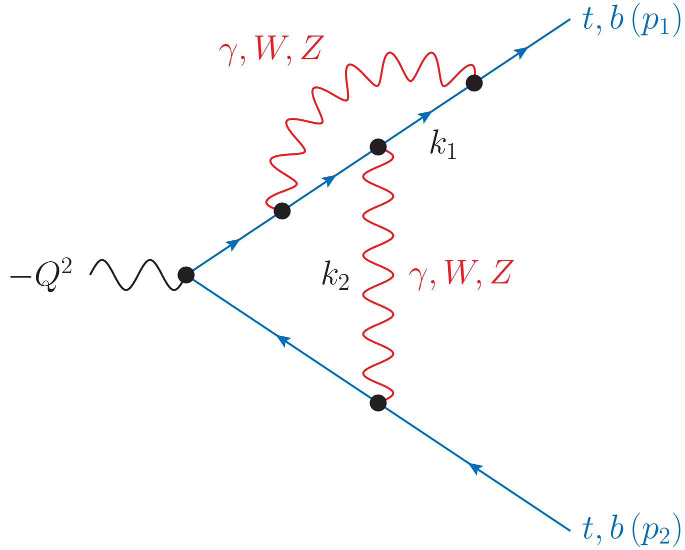
The expansion by regions has been applied to many complicated loop integrals. Example:

2-loop vertex integral in the high-energy limit

$\boxed{Q^2 \gg m_t^2}$ $\leadsto$ 9 relevant regions "$(k_1 - k_2)$":

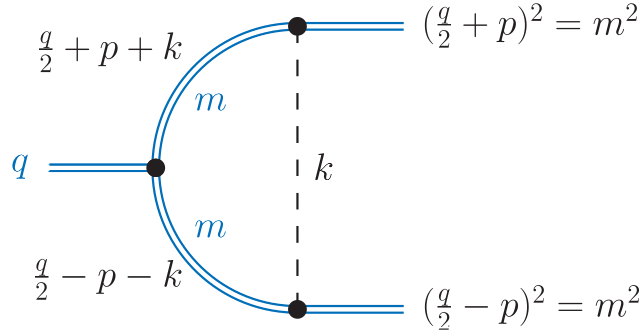
$(h-h)$, $(1c-h)$, $(h-2c)$, $(1c-1c)$, $(1c-2c)$,
$(2c-2c)$, $(us-2c)$, $(1c-2uc)$, $(2uc-2uc)$

$\hookrightarrow$ Next-to-leading-log result cross-checked with other methods.



Example with 3 regions: threshold expansion for heavy-particle pair production

Regions analyzed in Beneke, Smirnov, NPB 522, 321 (1998)



Centre-of-mass system: $(q^\mu)=(q_0,\vec{0})$, $(p^\mu)=(0,\vec{p})$

Close to threshold: $q^2 \approx (2m)^2 \Rightarrow$ $q^2 \gg |p^2|$ or $q_0 \gg |\vec{p}|$

$$F=\int \frac{Dk}{(k^2+q_0k_0-2\vec{p}\cdot\vec{k})\,(k^2-q_0k_0-2\vec{p}\cdot\vec{k})\,k^2}$$

- **hard** (h): $k_0,|\vec{k}|\sim q_0 \Rightarrow$ expand $\sum_j T_j^{(h)}$ in $D_h=\left\{k\in\mathbb{R}^d \,\middle|\, |k_0|\gg|\vec{p}| \text{ or } |\vec{k}|\gg|\vec{p}|\right\}$
- **soft** (s): $k_0,|\vec{k}|\sim|\vec{p}| \Rightarrow$ expand $\sum_j T_j^{(s)}$ in $D_s=\left\{k\in\mathbb{R}^d \,\middle|\, |\vec{k}|\lesssim|k_0|\lesssim|\vec{p}|\right\}$
- **potential** (p): $k_0\sim\frac{\vec{p}^{\,2}}{q_0}$, $|\vec{k}|\sim|\vec{p}| \Rightarrow \sum_j T_j^{(p)}$ in $D_p=\left\{k\in\mathbb{R}^d \,\middle|\, |k_0|\ll|\vec{k}|\lesssim|\vec{p}|\right\}$

$\hookrightarrow$ $D_h\cup D_s\cup D_p=\mathbb{R}^d$, $D_h\cap D_s=D_h\cap D_p=D_s\cap D_p=\emptyset$ [no explicit boundaries needed]

$\hookrightarrow$ The expansions $T_{j_1}^{(h)},T_{j_2}^{(s)},T_{j_3}^{(p)}$ commute with each other.

$\Rightarrow$ **Identity:** $$F=F^{(h)}+\underbrace{F^{(s)}}_{=0}+F^{(p)}-\Big(\underbrace{F^{(h,s)}}_{=0}+\underbrace{F^{(h,p)}}_{=0}+\underbrace{F^{(s,p)}}_{=0}\Big)+\underbrace{F^{(h,s,p)}}_{=0\text{ (scaleless)}}$$

with

$$F^{(h)}=-\frac{2e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{q^2}\left(\frac{4\mu^2}{q^2}\right)^{\epsilon}\sum_{j=0}^{\infty}\frac{(1+\epsilon)_j}{j!\,(1+2\epsilon+2j)}\left(-\frac{4p^2}{q^2}\right)^j$$

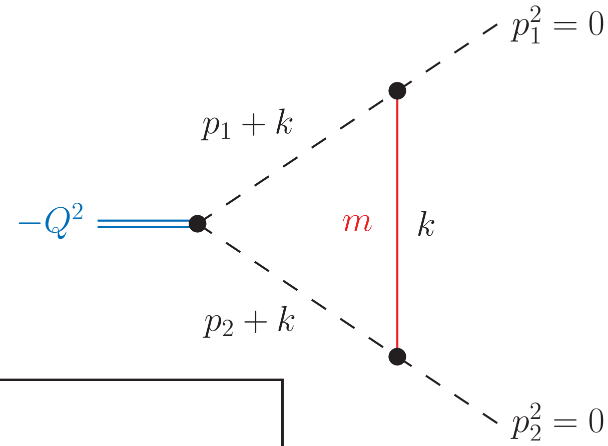
$$F^{(p)}=\frac{e^{\epsilon\gamma_E}\,\Gamma(\frac{1}{2}+\epsilon)\,\sqrt{\pi}}{2\epsilon\,\sqrt{q^2\,(p^2-i0)}}\left(\frac{\mu^2}{p^2-i0}\right)^{\epsilon}$$

[higher orders scaleless]

$F^{(h)}+F^{(p)}$ reproduces exact result

$$F=\frac{e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{2p^2}\left(\frac{\mu^2}{p^2-i0}\right)^{\epsilon}\times{}_2F_1\!\left(\frac{1}{2},1+\epsilon;\frac{3}{2};-\frac{q^2}{4p^2}-i0\right)$$

Sudakov form factor: non-commuting expansions



Light-cone coordinates: $2p_{1,2} \cdot k = Qk^\pm$, $p_{1,2} \cdot k_\perp = 0$

Sudakov limit: $-(p_1 - p_2)^2 = \boxed{Q^2 \gg m^2}$

With analytic regulator $\delta \to 0$:

$$F = \int \frac{Dk}{(k^+k^- - \vec{k}_\perp^2 + Qk^+)^{1+\delta}\,(k^+k^- - \vec{k}_\perp^2 + Qk^-)^{1-\delta}\,(k^+k^- - \vec{k}_\perp^2 - m^2)}$$

- **hard** (h): $k^+, k^-, |\vec{k}_\perp| \sim Q \Rightarrow D_h = \left\{k \in \mathbb{R}^d \middle| \vec{k}_\perp^2 \gg m^2\right\}$
- **1-collinear** $(1c)$: $k^+ \sim \frac{m^2}{Q}$, $k^- \sim Q$, $|\vec{k}_\perp| \sim m$
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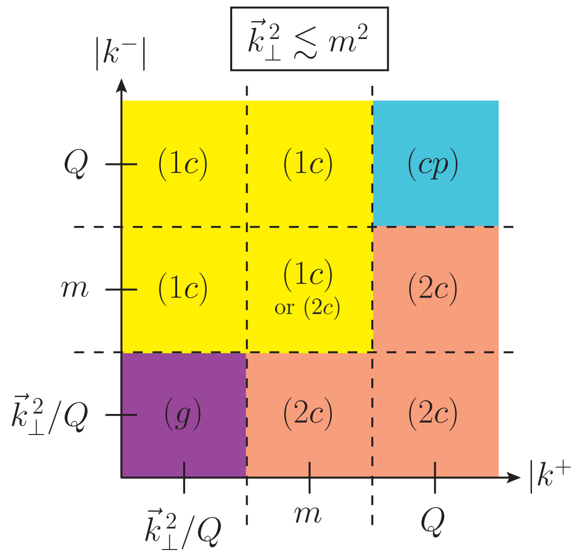

Most expansions commute, but $T^{(g)}T^{(cp)}I \neq T^{(cp)}T^{(g)}I$!

$\hookrightarrow$ Obtain only combinations of regions without both (g) and (cp)
+ extra terms which cancel at the integrand level:

$$F = \sum_{x'_1} F^{(x'_1)} - \sum_{\{x'_1, x'_2\} \neq \{g, cp\}} F^{(x'_1, x'_2)} + \sum_{\{x'_1, x'_2, x'_3\} \not\supset \{g, cp\}} F^{(x'_1, x'_2, x'_3)} - \left(F^{(h,1c,2c,g)} + F^{(h,1c,2c,cp)}\right)$$